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Comment on "Nontrivial Geometries: Bounds on the Curvature of the Universe"

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Abstract. The paper 0705.0332v1 seeks to study the effect of non-trivial spatial curvature in homogeneous and isotropic models. We note that the space considered is not homogeneous, and that the equations of motion used are inconsistent with the metric. Also, we explain why the spatial curvature of homogeneous and isotropic spacetimes always evolves like $1/a^2$, contrary to the central assumption of 0705.0332v1.

Introduction. The paper [1] seeks to study observational constraints on homogeneous and isotropic universes where the spatial curvature would evolve differently from the usual behaviour of being simply proportional to $1/a^2$, where a is the scale factor. However, the spacetime of [1] is not spatially homogeneous, the metric does not lead to the equations of motion used and is inconsistent with the energy-momentum tensor specified. Also, the spatial curvature in homogeneous and isotropic universes is always proportional to $1/a^2$. We explain these points below.

Homogeneity and isotropy. The analysis of [1] starts from the spacetime described by the metric (restoring the angular part)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a(t)^2 \frac{\mathrm{d}r^2}{1-K(t)r^2} + a(t)^2 r^2 (\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) \ , \tag{1}$$

which is like the usual Friedmann-Robertson-Walker (FRW) metric except that the curvature constant K has been replaced with a function of time $K(t)$. (In the notation of [1], $K(t) \equiv -x(a(t))/H_0^2$, where H_0 is the value of the Hubble parameter today.) However, once K is a function of time, the space described by (1) is no longer homogeneous, contrary to the assumption of [1]. (It is, of course, still manifestly isotropic.) The non-zero components of the Einstein tensor for the metric (1) read

$$\begin{aligned} G^t{}_t &= -3\frac{\dot{a}^2}{a^2} - 3\frac{K}{a^2} - \frac{\dot{a}}{a}\frac{\dot{K}r^2}{1-Kr^2} \\ G^r{}_r &= -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2} \end{aligned}$$

$$
\begin{aligned}
G^{\theta}{}_{\theta} = G^{\phi}{}_{\phi} &= -2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{K}{a^2} - \frac{3}{2}\frac{\dot{a}}{a}\frac{\dot{K}r^2}{1-Kr^2} - \frac{3}{4}\frac{\dot{K}^2 r^4}{(1-Kr^2)^2} - \frac{1}{2}\frac{\ddot{K}r^2}{1-Kr^2} \\
G_{tr} &= \frac{\dot{K}r}{1-Kr^2} \ . \qquad (2)
\end{aligned}
$$

From the fact that the spatial components of the Einstein tensor are not equal and the tr-component is not zero it is transparent that the space is not homogeneous unless K is constant. That this not just due to a choice of coordinates which would hide the homogeneity can be unambiguously established by evaluating the square of the Weyl tensor: it vanishes everywhere, and the space is homogeneous and isotropic, if and only if $\dot{K} = 0$.

The equation of motion. In [1], the Hamiltonian constraint is given as (writing it in terms of the energy densities rather than the Ω density parameters)

$$
3\frac{\dot{a}^2}{a^2} = 8\pi G_N(\rho_r + \rho_m) + \Lambda - 3\frac{K(t)}{a^2} \ , \qquad (3)
$$

where ρ_r and ρ_m are the energy density of radiation and matter, respectively, and Λ is the cosmological constant.

The equation (3) does not follow from applying the Einstein equation to the metric (1). The tt-component of the Einstein tensor (2), which one might naively equate with $-8\pi G_N\rho - \Lambda$ (where ρ is the total energy density), contains a term involving $\dot{K}$, which is not present in (3). However, because the coordinate system is not comoving, the tt-component of the energy-momentum tensor is not simply $-\rho$, so even including the $\dot{K}$-term would not lead to the correct equations. The energy-momentum tensor of an ideal fluid with energy density ρ, pressure p and velocity u^μ (which satisfies $u^\mu u_\mu = -1$) is

$$
T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \ , \qquad (4)
$$

with the obvious generalisation for the case of two ideal fluids. Thus, the tt-component of the Einstein tensor (2) should be equal to $\rho u^t u_t + p(1 + u^t u_t)$ (plus the contribution of the cosmological constant). This is equal to $-\rho$ only when $u^t u_t = -1$, i.e. when the coordinates are comoving. However, then $T_{tr} = 0$, which according to (2) implies $\dot{K} = 0$.

So, apart from the fact that the space is not homogeneous, the equations of motion given in [1] do not follow from, and are inconsistent with, the metric used in [1], unless K is constant.

(Every metric is the solution of the Einstein equation with some energy-momentum tensor. However, we may note that in addition to not being the solution for any homogeneous sources, the metric (1) is also not the solution for any combination of dust, radiation and cosmological constant, even if they are inhomogeneous; see e.g. [2].)

The evolution of the spatial curvature. The fact that the spatial curvature of a homogeneous and isotropic space is proportional to $1/a^2$ (i.e. that K is constant) is a

standard textbook result; see e.g. [3]. There is also a simple way to derive the behaviour of the spatial curvature from the Raychaudhuri equation without directly analysing the Riemann tensor of a homogeneous and isotropic three-space.

For a general rotationless ideal fluid, one can obtain the following local equations, without specifying a metric (see e.g. [4, 5]):

$$\dot{\theta} + \frac{1}{3}\theta^2 = -4\pi G_N(\rho + 3p) - 2\sigma^2 + \dot{u}^{\mu}{}_{;\mu} \tag{5}$$

$$\frac{1}{3}\theta^2 = 8\pi G_N \rho - \frac{1}{2}{}^{(3)}R + \sigma^2 \tag{6}$$

$$\dot{\rho} + \theta(\rho + p) = 0\ , \tag{7}$$

where dot stands for time derivative, θ is the expansion rate of the local volume element (in the FRW case, $\theta = 3\dot{a}/a$), σ^2 is the shear scalar, and ${}^{(3)}R$ is the spatial curvature. The acceleration equation (5) is known as the Raychaudhuri equation, and (6) is the Hamiltonian constraint. (As an aside, we note that for the metric (1), the volume expansion rate is not given by $3\dot{a}/a$, since the volume element contains the time-dependent factor $(1 - Kr^2)^{-1/2}$ in addition to a^3.)

The integrability condition between (5) and (6) is

$$\partial_t({}^{(3)}R) + \frac{2}{3}\theta\ {}^{(3)}R = 2\,\partial_t\sigma^2 + 4\theta\sigma^2 - \frac{4}{3}\theta\,\dot{u}^{\mu}{}_{;\mu}\ . \tag{8}$$

If the spacetime is homogeneous and isotropic, the shear and the acceleration are zero, so it immediately follows that the spatial curvature is proportional to $1/a^2$.

A heuristic way of obtaining this result is to note that the spatial curvature never contributes to the Raychaudhuri equation, while the contribution of the energy density and pressure of an ideal fluid is proportional to $\rho + 3p$. Treating the spatial curvature as an effective ideal fluid, it then follows that its effective equation of state is $p = -\rho/3$, which by (7) translates into $\rho \propto 1/a^2$. As the rigorous derivation above shows, this argumentation holds only in the FRW case, since in the presence of inhomogeneity and/or anisotropy the spatial curvature cannot be treated as an independently conserved ideal fluid, due to the coupling of the spatial curvature to shear and acceleration.

Inhomogeneous and/or anisotropic space. Since (8) shows that spatial curvature in an inhomogeneous and/or anisotropic space does not evolve as simply as in the FRW case, one can ask whether the *average* spatial curvature of an inhomogeneous and/or anisotropic space could behave as modelled in [1] (though this was not the way the non-trivial spatial curvature was motivated in [1]). For perturbations with wavelengths much larger than the Hubble scale, the answer is negative. If the local universe is smooth and only super-Hubble perturbations are present, the spatial curvature evolves like $1/a^2$, at least for spacetimes dominated by dust and/or a cosmological constant [6–8].

When perturbations on scales smaller than the Hubble scale are present, the average spatial curvature does indeed evolve in a non-FRW manner, and the departure from the $1/a^2$ scaling law is directly related to the non-FRW evolution of the average expansion rate. Then the effects of inhomogeneity and/or anisotropy contribute to the

Hamiltonian constraint, as is clear from (6), so one cannot simply plug in the non-trivial spatial curvature into the FRW equations. Instead, the evolution of the averages is governed by the Buchert equations [9,10], which include these effects (see [8] for further discussion). Determining the proper distance (and thus the luminosity distance and the angular diameter distance) in such a spacetime is a non-trivial problem even if the evolution of the scale factor is known, precisely because the way the spatial hypersurfaces are curved does not follow the simple FRW rule. For some work on evaluating the luminosity distance with a non-trivially evolving scale factor while neglecting the non-trivial evolution of the spatial curvature, see [11].

Summary. In conclusion, the metric introduced in [1] does not describe a homogeneous space, and the equation of motion used in [1] is inconsistent with the metric. The only exception is when the function $x(a(t))$ in the metric is constant. In this case, the FRW metric is recovered, and there is no "non-trivial geometry", which was the topic of [1].

Acknowledgments

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